Newton's inequality

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Friday, July 6, 2018

For $x_1, \ldots, x_n$, we define the symmetric sum s_k to be the coefficient of t^{n-k} in the polynomial $\prod_{i=1}^{n}(t+x_i)$ (see Viete's sums). We define the symmetric average d_k to be $s_k/\binom{n}{k}$.

Theorem 1. *For non-negative $x_1, \ldots, x_n$ and $0 < k < n$, $d_k^2 \geq d_{k-1}d_{k+1}$, with equality exactly when all the x_i are equal.*

Proof. We will proceed by induction on n.

For $n = 2$, the inequality just reduces to AM-GM inequality. Now suppose that for $n = m-1$ some positive integer $m \geq 3$ the inequality holds.

Let x_1, x_2, …, x_m be non-negative numbers and d_k be the symmetric averages of them. Let d'_k be the symmetric averages of x_1, …, x_{k-1}. Note that $d_k = \frac{n-k}{n}d'_k + \frac{k}{n}d'_{k-1}x_m$.

$$\begin{aligned}
d_{k-1}d_{k+1} =& \left(\frac{n-k+1}{n}d'_{k-1} + \frac{k-1}{n}d'_{k-2}x_m\right)\left(\frac{n-k-1}{n}d'_{k+1} + \frac{k+1}{n}d'_k x_m\right)\\
=& \frac{(n-k+1)(n-k-1)}{n^2}d'_{k-1}d'_{k+1} + \frac{(k-1)(n-k-1)}{n^2}d'_{k-2}d'_{k+1}x_m\\
&+ \frac{(n-k+1)(k+1)}{n^2}d'_{k-1}d'_k x_m + \frac{(k-1)(k+1)}{n^2}d'_{k-2}d'_k x_m^2\\
\leq& \frac{(n-k+1)(n-k-1)}{n^2}d'^2_k + \frac{(k-1)(n-k-1)}{n^2}d'_{k-2}d'_{k+1}x_m\\
&+ \frac{(n-k+1)(k+1)}{n^2}d'_{k-1}d'_k x_m + \frac{(k-1)(k+1)}{n^2}d'^2_{k-1}x_m^2\\
\leq& \frac{(n-k+1)(n-k-1)}{n^2}d'^2_k + \frac{(k-1)(n-k-1)}{n^2}d'_{k-1}d'_k x_m\\
&+ \frac{(n-k+1)(k+1)}{n^2}d'_{k-1}d'_k x_m + \frac{(k-1)(k+1)}{n^2}d'^2_{k-1}x_m^2\\
=& \frac{(n-k)^2}{n^2}d'^2_k + \frac{2(n-k)k}{n^2}d'_k d'_{k-1}x_m + \frac{k^2}{n^2}d'^2_{k-1}x_m^2 - \left(\frac{d_k}{n} - \frac{d_{k-1}x_m}{n}\right)^2\\
\leq& \left(\frac{n-k}{n}d'_k + \frac{k}{n}d'_{k-1}x_m\right)^2\\
=& d_k^2
\end{aligned}$$

□